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The Wigner Distribution Function With Minimum Spread

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The Wigner distribution function (WDF) with minimum quadratic spread corresponds to a Gaussian amplitude-modulated waveform with linear frequency-modulation. The optimum WDF is two-dimensional Gaussian and has contours of equal height which are identical to the penalty contours of the quadratic spread measure employed. An alternative measure of spread, involving an exponential reward for concentration, leads to identically the same optimum waveform and WDF. A generalization to a certain class of smoothed WDFs is also possible and is presented.

The sensitivity of the effective area of a smoothed WDF, to mismatch in shape factor and tilt in the time-frequency plane, is evaluated quantitatively.

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Short-Term Spectral Estimate
Smoothing
Two-Dimensional Convolution
Concentrated WDF

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

WDF	Wigner distribution function
t	time
f	frequency
$s(t)$	waveform analyzed, (1)
$u(t)$	weighting function, (1)
S_u	short-term spectral estimate, (1)
W_s	WDF of $s(t)$, (1)
W_u	WDF of $u(t)$, (1), (2)
$\circledast$	convolution, (1)
δ	delta function
$P(t,f)$	penalty measure, (3)
a,b,c	parameters of P, (3)
I	spread of WDF W_u, (4)

LIST OF SYMBOLS (Cont'd)

Q	real constant, (5)
$u'(t)$	derivative of $u(t)$ with respect to t, (10)
B	complex constant, (11)
$T(t)$	temporary auxiliary function, (13)
$u_o(t)$	optimum weighting, (20)
I_o	minimum spread, (21)
W_o	optimum WDF, (22)
D	generalized smoothing distribution, (23)
V_2	smoothing function, (23)
ν	frequency separation variable, (23), (24)
τ	time separation variable, (23), (24)
χ_u	complex ambiguity function of u, (24)
q_2	double Fourier transform of V_2, (25)
superscript ν	partial derivative with respect to ν, (27)
superscript τ	partial derivative with respect to τ, (27)
I_D	spread of distribution D, (30)
$R(t,f)$	reward function, (22A), (A-1)
V	reward value, (22B), (A-5)
$K(x,y)$	Hermitian kernel, (A-11), (A-12)
λ_n	eigenvalue, (A-12)
$\phi_n(x)$	eigenfunction, (A-12)
$He_n(x)$	Hermite polynomial, (A-27)
A	area of ellipse, figure C-1
β	tilt of ellipse, figure C-1
F	shape factor of ellipse, (C-4)

THE WIGNER DISTRIBUTION FUNCTION WITH MINIMUM SPREAD

INTRODUCTION

A number of advantageous features associated with smoothing a Wigner distribution function (WDF) were discussed in a recent report [1]. At that time, it was shown that the WDF with minimum quadratic spread, about the line $f = \beta_c t$ in the time-frequency plane, was a two-dimensional Gaussian function, when constraints of finite energy and mean-square duration were imposed [1, app. G]. However, a more appropriate measure of spread about the origin in the t,f plane is adopted here and minimized, yielding a unique waveform and corresponding WDF. Additionally, a reward measure for concentration is shown to yield identically the same optimum WDF.

An additional property of smoothing two-dimensional WDFs was also demonstrated; namely, if two Gaussian mountains are doubly-convolved with each other, the effective area of the result is greater than the sum of the two effective areas, unless the contours of both WDFs have the same tilt and ratio of major-to-minor axes [1, app. J]. A quantitative investigation of the effect of mismatch in these parameters on the effective area is conducted herein.

It is assumed that the reader is familiar with the content and approach of the earlier report; accordingly, this follow-on effort will be briefer and will not review the considerable history and background of the WDF.

MINIMUM QUADRATIC SPREAD

It was shown in [1, (102) and (106)] that the short-term spectral estimate is equal to the double convolution of the WDF of the waveform s(t) being analyzed with the WDF of the weighting u(t) employed. That is,

$$\left|S_u(t,f)\right|^2 \equiv \left|\int dt_1 \exp(-i2\pi f t_1)\ s(t_1)\ u^*(t - t_1)\right|^2 =$$

$$= \iint dt_1\ df_1\ W_s(t_1,f_1)\ W_u(t - t_1, f - f_1) = W_s(t,f) \overset{tf}{\circledast} W_u(t,f) \ , \qquad (1)$$

where $\circledast$ denotes convolution. Here,

$$W_u(t,f) = \int d\tau \exp(-i2\pi f\tau)\ u(t + \tfrac{\tau}{2})\ u^*(t - \tfrac{\tau}{2}) \qquad (2)$$

is the WDF of complex weighting u(t); a similar definition holds for WDF W_s. (Generalizations to non-Wigner smoothing functions for W_u are given in [1, app. F].)

Since the WDF W_s of waveform s(t) has some good energy localization properties (and some deleterious negative oscillations), it is desired that the smearing in the t,f plane, implied by convolution (1), be minimized. That is, we would like WDF W_u of weighting u(t) to be as concentrated as possible about the origin of the t,f plane. The ideal of an impulse, $\delta(t)\delta(f)$, is not a legal WDF, and must be discarded. Since the left-hand side of (1) can never be negative, we can be assured, by this smoothing procedure of two WDFs, that we will always get a physically-meaningful distribution in the t,f plane; that is, the smoothed distribution will always be non-negative for all t,f and have a volume equal to the energy of waveform s(t). For example, see [1, (111) et seq.].

PENALTY MEASURE AND SPREAD

In order to confine WDF W_u near the origin, we define a penalty measure which is zero at t,f = 0,0 and which increases quadratically with t and f. Namely, the penalty measure is

$$P(t,f) = a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf \ , \quad a,b,c \text{ real} \ , \tag{3}$$

and the corresponding spread of the WDF W_u is defined as

$$I = \iint dt\ df\ W_u(t,f)\ P(t,f) \ . \tag{4}$$

Contours of equal penalty in (3) are tilted ellipses in the t,f plane; these would be selected upon observation of a calculated WDF W_s of waveform s(t) in regions of interest, i.e., high activity.

Therefore, real constants a,b,c are presumed known. Define quantity

$$Q = a^2b^2 - c^2 \ . \tag{5}$$

Then, in order that penalty

$$P(t,f) > 0 \quad \text{for} \quad t,f \neq 0,0 \ , \tag{6}$$

it is necessary that

$$Q > 0 \ . \tag{7}$$

The property (6) was not satisfied by penalty function $(f - \beta_c t)^2$ in [1, app. G]; that function was zero all along the line $f = \beta_c t$, allowing the WDF to become impulsive there.

We also want WDF W_u in (1) and (4) to have unit volume, for two reasons. First of all, this will guarantee that the short-term spectral estimate on the left-hand side of (1) will have a volume equal to the signal energy, regardless of weighting u(t) employed. Secondly, without this volume constraint, u(t) and W_u would collapse to zero, giving a meaningless spread value of I = 0 in (4). Thus we require that

$$1 = \iint dt\ df\ W_u(t,f) = \int dt\ |u(t)|^2 . \tag{8}$$

Subject to this integral constraint, we want to minimize spread I in (4), and find the particular weighting u(t) and corresponding optimum WDF W_u. Notice that we are imposing no constraint of positivity on W_u.

DERIVATION OF SPREAD

Substitute (3) into (4) to get spread

$$I = \iint dt\ df\ W_u(t,f)\ (a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf) , \tag{9}$$

where WDF W_u is given in terms of u(t) according to (2). By using the results in [1, (G-4) and (G-5)], we can express (9) solely in the time domain as

$$I = a^2\int dt\ t^2\ |u(t)|^2 + b^2\int dt\ |u'(t)|^2 + 2c\int dt\ t\ \mathrm{Im}\{u'(t)\ u^*(t)\} =$$

$$= \int dt\ [a^2t^2|u(t)|^2 + b^2|u'(t)|^2 + ict\ u(t)\ u'^*(t) - ict\ u^*(t)\ u'(t)]. \tag{10}$$

For reasons to become apparent shortly, define complex constant

$$B = \frac{\sqrt{Q} + ic}{b^2} ; \tag{11}$$

then, by using (5), we find

$$b^2 |B|^2 = a^2 \ . \tag{12}$$

Now consider the quantity

$$T(t) \equiv b^2 \left| u'(t) + Bt\ u(t) \right|^2 =$$

$$= b^2 |u'(t)|^2 + b^2 |B|^2 t^2 |u(t)|^2 + b^2 Bt\ u(t)\ u'^*(t) + b^2 B^*\ t\ u^*(t)\ u'(t) =$$

$$= b^2 |u'(t)|^2 + a^2 t^2 |u(t)|^2 + (\sqrt{Q} + ic)t\ u(t)\ u'^*(t) +$$

$$+ (\sqrt{Q} - ic)t\ u^*(t)\ u'(t) \ . \tag{13}$$

Comparison of (10) and (13) immediately reveals that

$$\int dt\ T(t) = I + \sqrt{Q} \int dt\ t\ [u(t)\ u'^*(t) + u^*(t)\ u'(t)] \ . \tag{14}$$

We now integrate by parts, letting

$$U = t\ u(t),\ dV = dt\ u'^*(t) \ , \tag{15}$$

to find that

$$\int dt\ t\ u(t)\ u'^*(t) = - \int dt\ [u(t) + t\ u'(t)]\ u^*(t) =$$

$$= - \int dt\ |u(t)|^2 - \int dt\ t\ u'(t)\ u^*(t) \ . \tag{16}$$

We presume that $u(t)$ goes to zero at $t = \pm\infty$, consistent with energy constraint (8).

When (16) is employed in (14), there follows

$$\int dt\ T(t) = I - \sqrt{Q} \int dt\ |u(t)|^2 \ . \tag{17}$$

Thus the desired expression for spread I is given by (17) and (13) as

$$I = b^2 \int dt \left| u'(t) + Bt\, u(t) \right|^2 + \sqrt{Q} \int dt \left| u(t) \right|^2 . \tag{18}$$

This general result holds for any weighting u(t); it is obviously positive in all cases, since $Q > 0$.

OPTIMUM WEIGHTING

The last term in (18) cannot be altered; it is equal to $\sqrt{Q}$, as seen by reference to constraint (8). Furthermore, the minimum value for the remaining term in (18) is zero and is obtained for weighting u(t) which satisfies the differential equation

$$u'(t) + B\, t\, u(t) = 0 \quad \text{for all } t . \tag{19}$$

The <u>only</u> solution to (19) is

$$u_o(t) = A \exp(-\tfrac{1}{2} Bt^2) \quad \text{for all } t , \tag{20}$$

where complex constant A is chosen for unit energy, and B is given by (11). That is, $u_o(t)$ has Gaussian amplitude-modulation and linear frequency-modulation. The phase of A is ambiguous.

The resultant minimum value of spread I in (18) is obviously

$$I_o = \sqrt{Q} = \sqrt{a^2 b^2 - c^2} , \tag{21}$$

where we employed (5). It is always positive, as seen by reference to requirements (6) and (7).

OPTIMUM WDF

The WDF corresponding to optimum weighting (20) is obtained by substitution in (2), and use of [1, (H-17) and (H-18)], as

$$W_o(t,f) = 2 \exp\left[-\frac{a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf}{\sqrt{Q}}\right] . \tag{22}$$

The area of the contour ellipse at the 1/e relative level is 1/2 in the t,f plane, as expected.

Observe that the numerator of the exp in (22) is identically the quadratic penalty function P(t,f) imposed in (3). That is, the contours of optimum WDF (22) are identical to the contours of equal penalty of P(t,f) in (3). This result is intuitively satisfying: the optimum WDF packs as much volume inside a given penalty contour as possible, to the extent that the resultant WDF values are equal all along that given penalty contour.

Observe also, that although positivity of the WDF W_u was not imposed as a constraint in the minimization of spread I in (4) or (9), the resultant optimum WDF in (22) is, in fact, everywhere positive. Although the optimum weighting (20) has an ambiguous phase, the optimum WDF has no ambiguity; there is a unique optimum WDF, namely (22).

ALTERNATIVE REWARD MEASURE

Instead of penalizing the spread of WDF W_u about the origin in t,f space, we could alternatively utilize a measure which rewards concentration about t,f = 0,0. In particular, consider reward function

$$R(t,f) = \exp[-a^2t^2 - 4\pi^2b^2f^2 - 4\pi ctf] \tag{22a}$$

and reward value

$$V = \iint dt\, df\, R(t,f)\, W_u(t,f) \tag{22b}$$

for WDF W_u. The origin value of R(t,f) is 1; in order for R(t,f) to decay to zero as t and/or f tend to infinity, we must have condition (7) satisfied again. Notice that the contours of equal reward are ellipses in the t,f plane.

The maximization of reward value V, subject to volume constraint (8) on W_u, is conducted in appendix A. It is shown there that the optimum weighting is again (20), and that the optimum WDF is (22). The maximum value of reward V is

$$V_{max} = \frac{1}{1+\sqrt{Q}} = \frac{1}{1+\sqrt{a^2b^2 - c^2}}. \tag{22c}$$

More general results, for arbitrary reward functions R(t,f) in (22b), are presented in appendix A.

GENERALIZATION TO SMOOTHED WDF

A general class of distributions* has been presented in [2, (1.7) and (1.8)]. In current notation, that class is given by [1, (F-1)] as

$$D(t,f) \equiv W_u(t,f) \overset{tf}{\circledast} V_2(t,f) =$$

$$= \iint d\nu \, d\tau \, \exp(i2\pi\nu t - i2\pi f\tau) \, \chi_u(\nu,\tau) \, q_2(\nu,\tau) \, , \tag{23}$$

where WDF W_u is given by (2), and $V_2(t,f)$ is a general two-dimensional smoothing function. The complex ambiguity function of u(t) is

$$\chi_u(\nu,\tau) = \int dt \, \exp(-i2\pi\nu t) \, u(t + \frac{\tau}{2}) \, u^*(t - \frac{\tau}{2}) \, , \tag{24}$$

while

$$q_2(\nu,\tau) = \iint dt \, df \, \exp(-i2\pi\nu t + i2\pi f\tau) \, V_2(t,f) \tag{25}$$

is a double Fourier transform of the smoothing function V_2. Observe that if there is no smoothing, then

$$V_2(t,f) = \delta(t) \, \delta(f)$$

$$q_2(\nu,\tau) = 1 \text{ for all } \nu,\tau$$

$$D(t,f) = W_u(t,f) \, . \tag{26}$$

---

*This section is based upon a suggestion by Leon Cohen, Hunter College, New York, NY, that the optimum WDF results here actually apply to a wider class of distributions.

Now it is shown in appendix B that the following second moments of generalized smoothing distribution D can be expressed in terms of derivatives of χ_u and q_2 at the origin:

$$\iint dt\, df\, t^2\, D(t,f) = -\frac{1}{4\pi^2}\,[\chi_u^{\nu\nu}(0,0)\; q_2(0,0) + 2\chi_u^{\nu}(0,0)\; q_2^{\nu}(0,0) +$$

$$+ \chi_u(0,0)\; q_2^{\nu\nu}(0,0)] \;,$$

$$\iint dt\, df\, t\, f\, D(t,f) = \frac{1}{4\pi^2}\,[\chi_u^{\tau\nu}(0,0)\; q_2(0,0) + \chi_u^{\tau}(0,0)\; q_2^{\nu}(0,0) +$$

$$+ \chi_u^{\nu}(0,0)\; q_2^{\tau}(0,0) + \chi_u(0,0)\; q_2^{\nu\tau}(0,0)] \;,$$

$$\iint dt\, df\, f^2\, D(t,f) = -\frac{1}{4\pi^2}\,[\chi_u^{\tau\tau}(0,0)\; q_2(0,0) + 2\chi_u^{\tau}(0,0)\; q_2^{\tau}(0,0) +$$

$$+ \chi_u(0,0)\; q_2^{\tau\tau}(0,0)] \;. \tag{27}$$

Here, for example, superscript ν denotes a partial derivative with respect to ν, which is then evaluated at the origin $\nu,\tau = 0,0$.

If follows immediately that if origin value

$$q_2(0,0) = 1 \;, \tag{28}$$

and if the five origin derivatives

$$q_2^{\nu}(0,0) = q_2^{\tau}(0,0) = q_2^{\nu\nu}(0,0) = q_2^{\nu\tau}(0,0) = q_2^{\tau\tau}(0,0) = 0 \;, \tag{29}$$

then (27) reduces to the moments that would have resulted from employing the no-smoothing result (26) in (27). Thus, distributions D(t,f) resulting from (23), with properties (28) and (29) for q_2, have the same second moments

as the WDF $W_u(t,f)$. Hence, the spread I_D of distribution $D(t,f)$ is given by (see (9))

$$I_D = \iint dt\, df\, D(t,f)\, (a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf) =$$

$$= \iint dt\, df\, W_u(t,f)\, (a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf) = I \ , \qquad (30)$$

which is exactly the spread I of WDF $W_u(t,f)$. That is, smoothed distribution $D(t,f)$ in (23) has the same spread as WDF $W_u(t,f)$, when smoothing function $V_2(t,f)$ (actually transform q_2) satisfies the properties in (28) and (29). Notice that these properties are considerably less restrictive than requiring

$$q_2(\nu,0) = q_2(0,\tau) = 1 \quad \text{for all } \nu,\tau , \qquad (31)$$

which arises when one is interested in maintaining the marginals [2, (1.6)].

We must also observe from (23) that the volume under generalized smoothing distribution D is equal to the product of the volume under W_u and the volume under V_2. But the latter quantity is unity, by virtue of (28). What all this means is, that if we minimize spread I_D in (30), subject to a unit volume constraint on D, the end result is precisely (18) and (20), and the optimum WDF W_u is again given by (22). The corresponding distribution D is obtained by substitution of (22) into (23) and specification of the complete V_2 or q_2 functions. The properties in (28) and (29) are not sufficient to completely specify q_2 or D; all that is specified by (28) and (29) are the second order moments of D in (27). It should also be noted that all the conditions in (29) cannot be met by the general tilted Gaussian q_2 function employed in [1, (F-9) and sequel to (F-12)].

SENSITIVITY TO MISMATCH

In [1, app. J], it was shown that if two Gaussian mountains are doubly-convolved in x,y space, the effective area A_3 of the resultant is greater than the sum of the individual areas, except when the two elliptical contours have the same tilt and the same ratio of major-to-minor axis (shape factor). Here, we wish to investigate, quantitatively, the increase in effective area above the minimum value, when the tilt and shape factors are not at their optimum values. This situation can arise when observation of WDF W_s of waveform s(t) is contaminated, in a particular region of interest in the t,f plane, by interference effects, thereby making estimation of the tilt and the shape factor of the elliptical contours somewhat inaccurate.

The general situation is considered mathematically in appendix C. Ellipse 1 has

$$\text{area } A_1, \text{ tilt } \beta_1, \text{ shape factor } F_1 , \tag{32}$$

while ellipse 2 has

$$\text{area } A_2, \text{ tilt } \beta_2, \text{ shape factor } F_2 . \tag{33}$$

The ratio

$$\frac{A_3}{A_1 + A_2} \tag{34}$$

is presented in (C-13) in terms of a number of auxiliary quantities.

The initial example we consider is where ellipse 1 has seven different areas, namely

$$A_1 = .5,1,2,3,4,5,6 \qquad \beta_1 = \frac{\pi}{4} \qquad F_1 = 2 \ . \tag{35}$$

The tilt is fixed at $\pi/4$ radians and the shape factor at 2. On the other hand, ellipse 2 has

$$A_2 = 2 \qquad \beta_2 = -\frac{\pi}{4} \text{ to } \frac{\pi}{4} \qquad F_2 = 2 \ . \tag{36}$$

That is, the shape factor is perfect at $F_2 = F_1 = 2$, but the tilt is swept over a $\pi/2$ range (greater discrepancies than $\pi/2$ lead to obvious periodicities and symmetries centered about $\beta_2 = \beta_1$ as well as about $\beta_2 = \beta_1 \pm \pi/2$ and about $\beta_2 = \beta_1 + \pi$). The situation under investigation is depicted in figure 1, where ellipse 2 is dotted.

The effect of mismatch in tilt is presented quantitatively in figure 2. As expected, ratio (34) is 1 at $\beta_2 = \beta_1 = \pi/4$, regardless of area A_1. The most degradation (upper-most curve) is realized for $A_1 = 2$, i.e., when the areas of the two ellipses are equal. The maximum increase in area is only 25 percent, when β_2 is off by $\pi/2$ radians; however, if the shape factor is significantly larger than 1, the sensitivity to the tilt would be much greater, as figure 1 shows.

The final example utilizes the exact same parameter values as (35) for ellipse 1, while ellipse 2 has

$$A_2 = 2 \qquad \beta_2 = \frac{\pi}{4} \qquad F_2 = 2 \text{ to } 6. \tag{37}$$

Now the tilt is perfect at $\beta_2 = \beta_1 = \pi/4$, but the shape factor F_2 varies above the best value of 2. The situation is depicted in figure 3, where ellipse 2 is again dotted.

Ratio (34) is plotted in figure 4 versus the shape factor F_2. Again, the upper-most curve corresponds to the case where $A_1 = A_2 = 2$. There is no need to compute ratio (34) for $F_2 < F_1 = 2$, because the values for $F_2 = F_1 r$ are the same as those for $F_2 = F_1/r$. Additional cases of interest can be investigated by use of the program listed in appendix C.

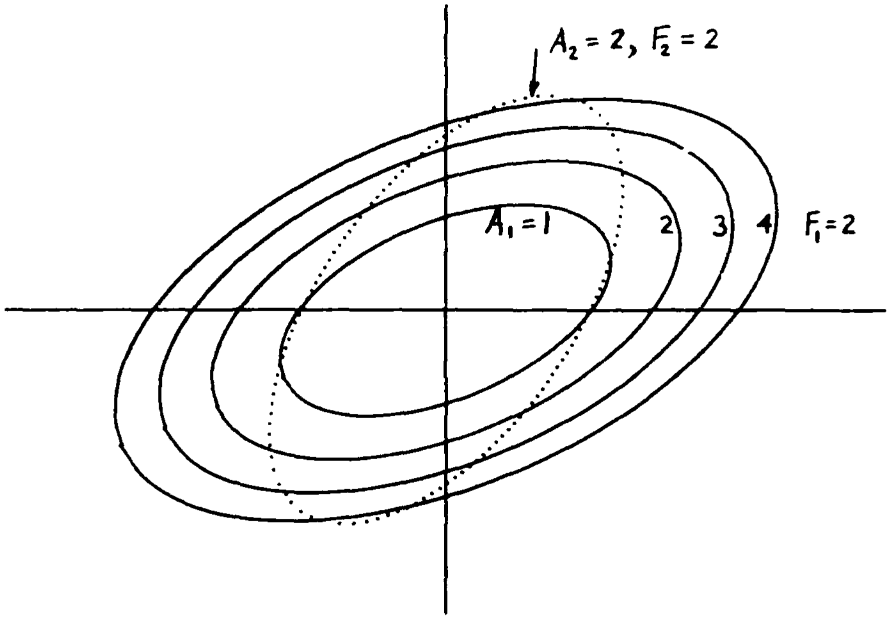

Figure 1. Contour Ellipses for Mismatched Tilt

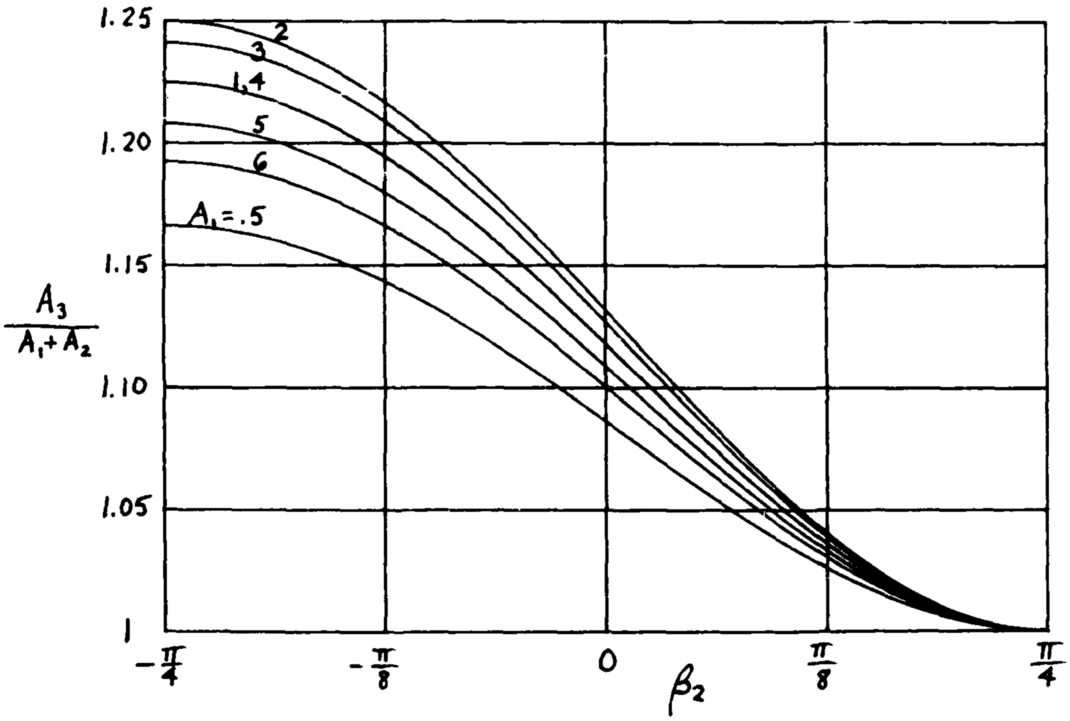

Figure 2. Area Ratio (34) for $F_1 = 2$, $F_2 = 2$, $A_2 = 2$, $\beta_1 = \pi/4$

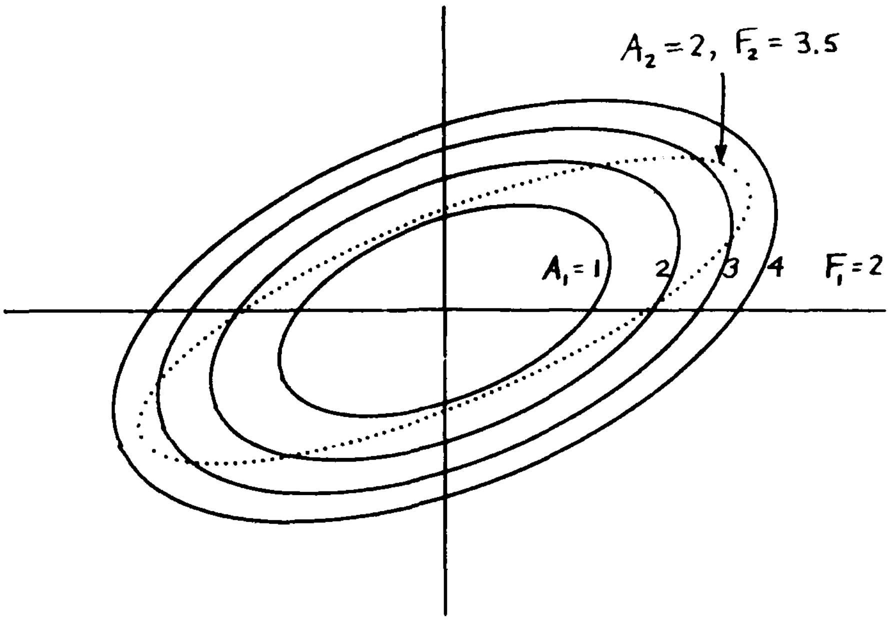

Figure 3. Contour Ellipses for Mismatched Shape Factor

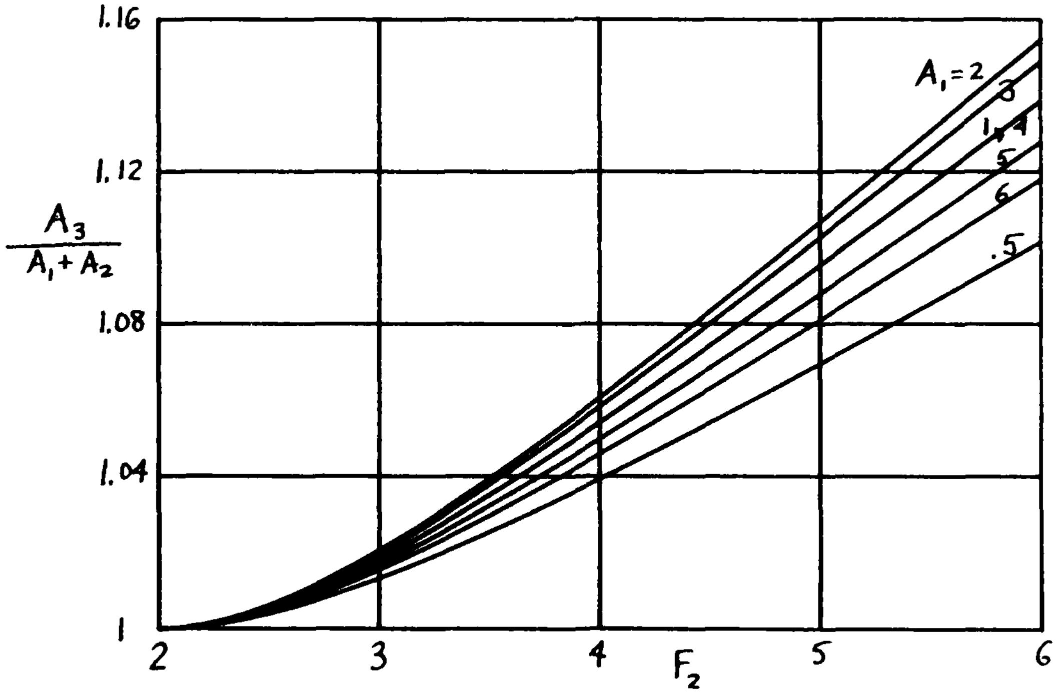

Figure 4. Area Ratio (34) for $F_1 = 2$, $A_2 = 2$, $\beta_1 = \pi/4$, $\beta_2 = \pi/4$

SUMMARY

The most compact WDF W_u that can be used for two-dimensional smoothing of a measured WDF W_s is a Gaussian function in two variables, when the measure of spread is quadratic in the time and frequency variables t and f, or the reward measure is exponential. Furthermore, this two-dimensional convolution guarantees a non-negative modified distribution, since the result is equivalent to a short-term spectral estimate. Extensions to a particular class of generalized distributions yields the same optimum WDF. The corresponding waveform has Gaussian amplitude modulation and linear frequency-modulation.

The additional smearing caused by mismatched smoothing functions to the true parameters of a measured WDF has been investigated numerically for a few examples, and found not to be overly sensitive to the exact values. However, the multitude of parameters has prevented simplification of the area spread factor; accordingly, a program allowing calculation of particular cases is included to allow for further investigation.

The WDFs for the Hermite functions of order n are given in closed form, in terms of a Laguerre polynomial of order n. This result is extended to cross-WDFs in appendix A; in this manner, we can investigate the WDF of an arbitrary waveform when expanded in a weighted sum of Hermite functions, including linear frequency-modulation.

APPENDIX A. MAXIMIZATION OF REWARD VALUE

We want to find that WDF, $W_u(t,f)$, which is maximally concentrated about the origin in t,f space, where the measure of reward for concentration is

$$R(t,f) = \exp[-a^2t^2 - 4\pi^2b^2f^2 - 4\pi ctf] \ , \quad a,b,c \text{ real} \ . \qquad \text{(A-1)}$$

Thus, the maximum reward occurs at the origin,

$$R(0,0) = 1 \ , \qquad \text{(A-2)}$$

and the contours of equal reward are ellipses in the t,f plane. In order for R(t,f) to decay to zero as t and/or f tend to infinity, we must have

$$Q > 0 \ , \qquad \text{(A-3)}$$

where

$$Q = a^2b^2 - c^2 \ . \qquad \text{(A-4)}$$

The reward value associated with WDF W_u is the real quantity

$$V = \iint dt\, df\, R(t,f)\, W_u(t,f) \ , \qquad \text{(A-5)}$$

which we wish to maximize, where

$$W_u(t,f) = \int d\tau \exp(-i2\pi f\tau)\, u(t + \tfrac{\tau}{2})\, u^*(t - \tfrac{\tau}{2}) \qquad \text{(A-6)}$$

in terms of weighting u(t). We must constrain the volume of W_u, in order that V in (A-5) not tend to infinity as u(t) is simply increased in level. Thus, we have integral constraint

$$1 = \iint dt\, df\, W_u(t,f) = \int dt\, |u(t)|^2 \ . \qquad \text{(A-7)}$$

ALTERNATIVE FORM FOR V

If we substitute (A-6) in (A-5), there follows

$$V = \iint dt\, d\tau\; r(t,\tau)\; u(t + \tfrac{\tau}{2})\; u^*(t - \tfrac{\tau}{2}) \;, \tag{A-8}$$

where

$$r(t,\tau) = \int df\; \exp(-i2\pi f\tau)\; R(t,f) \;. \tag{A-9}$$

A more useful alternative form for (A-8) is

$$V = \iint dx\, dy\; K(x,y)\; u(x)\; u^*(y) \;, \tag{A-10}$$

where kernel

$$K(x,y) = r\left(\frac{x+y}{2} \;,\; x-y\right) \;. \tag{A-11}$$

EIGENFUNCTIONS OF K

In this and the following subsection, kernel K is Hermitian, but otherwise arbitrary; it is not limited to form (A-11) with (A-9) and (A-1). Suppose $\{\lambda_n\}$ and $\{\phi_n\}$ are the eigenvalues and eigenfunctions of kernel K; i.e.,

$$\int dx\; K(x,y)\; \phi_n(x) = \lambda_n\; \phi_n(y) \quad \text{for } n = 0,1,2,\; \ldots \;, \tag{A-12}$$

where $\lambda_0 \geq \lambda_1 \geq \lambda_2 \; \ldots$, and

$$\int dx\; \phi_n^*(x)\; \phi_m(x) = \delta_{nm} \;. \tag{A-13}$$

Then the kernel can be expanded according to

$$K(x,y) = \sum_{n=0}^{\infty} \lambda_n \, \phi_n^*(x) \, \phi_n(y) \; . \tag{A-14}$$

Also, there follows immediately

$$\iint dx \, dy \, K(x,y) \, \phi_n(x) \, \phi_n^*(y) = \lambda_n \; . \tag{A-15}$$

EXPANSION OF u

Suppose we expand weighting u in a series of eigenfunctions of Hermitian kernel K:

$$u(x) = \sum_{n=0}^{\infty} g_n \, \phi_n(x) \; , \quad \text{where } g_n = \int dx \, u(x) \, \phi_n^*(x) \; . \tag{A-16}$$

Then general reward expression V in (A-10) becomes

$$V = \int dy \, u^*(y) \int dx \, K(x,y) \sum_{n=0}^{\infty} g_n \, \phi_n(x) =$$

$$= \sum_{n=0}^{\infty} g_n \int dy \, u^*(y) \, \lambda_n \, \phi_n(y) = \sum_{n=0}^{\infty} \left|g_n\right|^2 \lambda_n \; , \tag{A-17}$$

where we used (A-12) and (A-16). At the same time, the energy of u in (A-16) is

$$E_u = \int dx \, \left|u(x)\right|^2 = \sum_{n=0}^{\infty} \left|g_n\right|^2 \; . \tag{A-18}$$

Now if the energy E_u of u is constrained at 1, as in (A-7), then the best choice of coefficients $\{g_n\}$ to maximize V in (A-17) is, since $\lambda_o \geq \lambda_1 \geq \lambda_2$ ..., obviously

$$|g_o| = 1 \quad \text{and} \quad g_n = 0 \quad \text{for} \quad n \geq 1 . \tag{A-19}$$

That is, the optimum weighting is

$$u_o(x) = \phi_o(x) \exp(i\Theta) , \tag{A-20}$$

where constant Θ is arbitrary, while the maximum reward is

$$V_{max} = \lambda_o . \tag{A-21}$$

That is, the zero-th order eigenvalue and eigenfunction of general Hermitian kernel K in (A-12) are the solutions to the problem of interest here, namely maximization of reward value V in (A-10) by choice of weighting u. For a general kernel, a recursive numerical procedure could be employed on (A-12) to determine λ_o and ϕ_o, if desired.

The formulation in these last two subsections is actually general enough to cover the earlier penalty function considered in (3) et seq. The only difference is that the eigenvalues $\{\lambda_n\}$ now <u>increase</u> with n, and we must select the eigenfunction corresponding to the minimum eigenvalue, in order to realize the least penalty. This approach is the subject of appendix D.

SPECIAL CASE OF EXPONENTIAL REWARD

We now specialize the general results of the previous two subsections to the reward function (A-1). Substitution in (A-9) and use of (A-4) yields

$$r(t,\tau) = \frac{1}{2\sqrt{\pi}\, b} \exp\left[-\frac{Qt^2 + \frac{1}{4}\tau^2 - ict\tau}{b^2}\right]; \tag{A-22}$$

compare [1, (F-9) and (F-12)]. Then (A-11) immediately gives Hermitian kernel

$$K(x,y) = \frac{1}{2\sqrt{\pi}\, b} \exp\left[-\frac{x^2 D^* + y^2 D + 2xy(Q-1)}{4b^2}\right], \tag{A-23}$$

where

$$D = Q + 1 + i2c \; . \tag{A-24}$$

At this point, we refer to Mehler's expansion [3, (67)] to obtain (after some labor)

$$K(x,y) = \sum_{n=0}^{\infty} \lambda_n \, \phi_n^*(x) \, \phi_n(y) \; , \tag{A-25}$$

where

$$\lambda_n = \frac{(1 - \sqrt{Q})^n}{(1 + \sqrt{Q})^{n+1}} \; , \tag{A-26}$$

$$\phi_n(x) = A \exp\left[-\frac{x^2}{2}\,\frac{\sqrt{Q} + ic}{b^2}\right] \frac{1}{\sqrt{n!}}\, He_n\left(2^{1/2}\, Q^{1/4}\, x/b\right), \tag{A-27}$$

and

$$|A|^2 = \frac{Q^{1/4}}{\sqrt{\pi}\, b} \; . \tag{A-28}$$

The function $He_n(x)$ is the Hermite polynomial [4, 22.2.15]. It is easily verified that (A-27) satisfies orthonormality relation (A-13).

OPTIMUM WEIGHTING

Since $Q > 0$ by (A-3), the eigenvalues in (A-26) satisfy $\lambda_0 > \lambda_1 > \lambda_2 \ldots$. Therefore, the maximum reward is

$$V_{max} = \lambda_0 = \frac{1}{1 + \sqrt{Q}} = \frac{1}{1 + \sqrt{a^2 b^2 - c^2}}, \tag{A-29}$$

and the corresponding weighting is

$$u_0(t) = \phi_0(t) = A \exp\left[-\frac{t^2}{2} \frac{\sqrt{Q} + ic}{b^2}\right] \tag{A-30}$$

from (A 27) and (A-28). This is identical to (20) combined with (11). Therefore the optimum WDF is again (22) for reward measure (A-1), as well as penalty measure (3). The waveform in (A-30) has Gaussian amplitude modulation and linear frequency-modulation.

HIGHER-ORDER HERMITE FUNCTIONS

For $n > 0$, the reward values $\{\lambda_n\}$ in (A-26) are all less than optimum value λ_0. We have succeeded in obtaining these explicit values without having to evaluate the WDFs of the corresponding Hermite waveforms in (A-27). We now rectify this situation. The WDF of $\phi_n(t)$ in (A-27) is given by integral

$$W_n(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, \phi_n\left(t + \frac{\tau}{2}\right) \phi_n^*\left(t - \frac{\tau}{2}\right) =$$

$$= \frac{|A|^2}{n!} \int d\tau \, \exp\left[-i2\pi f\tau - \frac{1}{2} B\left(t + \frac{\tau}{2}\right)^2 - \frac{1}{2} B^*\left(t - \frac{\tau}{2}\right)^2\right] *$$

$$* \, He_n\left(F\left(t + \frac{\tau}{2}\right)\right) He_n\left(F\left(t - \frac{\tau}{2}\right)\right), \qquad \text{(A-31)}$$

where

$$B = \frac{\sqrt{Q} + ic}{b^2}, \quad F = \frac{2^{1/2} Q^{1/4}}{b}, \quad |A|^2 = \frac{Q^{1/4}}{\sqrt{\pi}\, b}. \qquad \text{(A-32)}$$

Now a more general integral result already exists in closed form; from [5, p. 292, (30)], we have, in a form more useful for present purposes,

$$\int dx \, \exp(-\frac{1}{2} x^2 + ax) \, He_m(b + x) \, He_n(b - x) =$$

$$= \sqrt{2\pi} \, (-1)^m \, m! \, (b - a)^{n-m} \, L_m^{(n-m)}(b^2 - a^2) \, \exp(a^2/2) \quad \text{for } m \le n, \qquad \text{(A-33)}$$

where $L_m^{(\alpha)}(x)$ is the generalized Laguerre polynomial [4, 22.2.12]. When (A-33) is used on (A-32), there follows, for the WDF of waveform $\phi_n(t)$ in (A-27), the compact result

$$W_n(t,f) = (-1)^n \, 2 \, L_n(2U) \, \exp(-U), \qquad \text{(A-34)}$$

where

$$U = \frac{a^2 t^2 + 4\pi^2 b^2 f^2 + 4\pi c t f}{\sqrt{Q}}. \qquad \text{(A-35)}$$

This result reduces to (22) for n = 0. Again, contours of equal values of the WDF are ellipses in the t,f plane.

CROSS-WDFs

Suppose a general waveform u(t) is expanded in a set of orthonormal Hermite functions with linear frequency-modulation (α positive real, β real)

$$\phi_n(t) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp[-\tfrac{1}{2}(\alpha + i\beta)t^2]\, He_n(\sqrt{2\alpha}\, t)/\sqrt{n!} \,, \qquad \text{(A-36)}$$

according to

$$u(t) = \sum_{n=0}^{\infty} u_n\, \phi_n(t) \ . \qquad \text{(A-37)}$$

Then the WDF of u(t) becomes

$$W_u(t,f) = \int d\tau \exp(-i2\pi f\tau)\, u(t + \tfrac{\tau}{2})\, u^*(t - \tfrac{\tau}{2}) =$$

$$= \sum_{m,n=0}^{\infty} u_m\, u_n^*\, W_{mn}(t,f) \ , \qquad \text{(A-38)}$$

where cross-WDF

$$W_{mn}(t,f) = \int d\tau \exp(-i2\pi f\tau)\, \phi_m(t + \tfrac{\tau}{2})\, \phi_n^*(t - \tfrac{\tau}{2}) \ . \qquad \text{(A-39)}$$

When (A-36) is substituted in (A-39), and (A-33) is utilized, the cross-WDF can be expressed as

$$W_{mn}(t,f) = 2(-1)^m \sqrt{\frac{m!}{n!}}\, z^{n-m}\, L_m^{(n-m)}\left(|z|^2\right) \exp\left(-|z|^2/2\right) \ \text{for } m \le n \ , \qquad \text{(A-40)}$$

where

$$z = \sqrt{\frac{2}{\alpha}}\, [\alpha t + i(2\pi f + \beta t)] \ ,$$

$$|z|^2 = \frac{2}{\alpha}\, [(\alpha^2 + \beta^2)t^2 + 4\pi^2 f^2 + 4\pi\beta t f] \ . \qquad \text{(A-41)}$$

These results generalize [6, pp. 456-7] and [7, p. 547].

The origin value of (A-40) is

$$W_{mn}(0,0) = 2(-1)^m \delta_{mn} , \tag{A-42}$$

consistent with unit energy of $\phi_n(t)$ and their even or odd character. The cross-WDF in (A-40) is a function only of the three variables m,n,z, where z is the complex combination in (A-41). The parameters α and β in (A-36) are perfectly general; when they are specialized to match (A-27), and when we set m = n, then (A-40) reduces to (A-34). Equations (A-38) and (A-40) afford a direct calculation of the WDF of a general waveform u(t), once the coefficients are determined by

$$u_n = \int dt\ u(t)\ \phi_n^*(t) . \tag{A-43}$$

APPENDIX B. MOMENTS OF DISTRIBUTION D

The generalized smoothing distribution D is given by (23) in terms of a double Fourier transform of product

$$P(\nu,\tau) = \chi_u(\nu,\tau)\; q_2(\nu,\tau) \; . \qquad \text{(B-1)}$$

Therefore, the inverse relation is

$$P(\nu,\tau) = \iint dt\; df\; \exp(-i2\pi\nu t + i2\pi f\tau)\; D(t,f) \; . \qquad \text{(B-2)}$$

If we let superscript ν denote a partial derivative with respect to ν, there immediately follows from (B-2),

$$\begin{aligned}
P(0,0) &= \iint dt\; df\; D(t,f) \\
P^{\nu}(0,0) &= -i2\pi \iint dt\; df\; t\; D(t,f) \\
P^{\tau}(0,0) &= i2\pi \iint dt\; df\; f\; D(t,f) \\
P^{\nu\nu}(0,0) &= -4\pi^2 \iint dt\; df\; t^2\; D(t,f) \\
P^{\nu\tau}(0,0) &= 4\pi^2 \iint dt\; df\; t\; f\; D(t,f) \\
P^{\tau\tau}(0,0) &= -4\pi^2 \iint dt\; df\; f^2\; D(t,f) \; .
\end{aligned} \qquad \text{(B-3)}$$

When these relations are written out explicitly in terms of χ_u and

q_2, according to (B-1), we find that the moments of D are

$$\iint dt\, df\, D(t,f) = \chi_u(0,0)\; q_2(0,0)$$

$$\iint dt\, df\, t\, D(t,f) = \frac{i}{2\pi}\,[\chi_u^{\nu}(0,0)\; q_2(0,0) + \chi_u(0,0)\; q_2^{\nu}(0,0)]$$

$$\iint dt\, df\, f\, D(t,f) = \frac{-i}{2\pi}\,[\chi_u^{\tau}(0,0)\; q_2(0,0) + \chi_u(0,0)\; q_2^{\tau}(0,0)]$$

$$\iint dt\, df\, t^2\, D(t,f) = -\frac{1}{4\pi^2}\,[\chi_u^{\nu\nu}(0,0)\; q_2(0,0) + 2\chi_u^{\nu}(0,0)\; q_2^{\nu}(0,0) +$$

$$+ \chi_u(0,0)\; q_2^{\nu\nu}(0,0)]$$

$$\iint dt\, df\, t\, f\, D(t,f) = \frac{1}{4\pi^2}\,[\chi_u^{\nu\tau}(0,0)\; q_2(0,0) + \chi_u^{\tau}(0,0)\; q_2^{\nu}(0,0) +$$

$$+ \chi_u^{\nu}(0,0)\; q_2^{\tau}(0,0) + \chi_u(0,0)\; q_2^{\nu\tau}(0,0)]$$

$$\iint dt\, df\, f^2\, D(t,f) = -\frac{1}{4\pi^2}\,[\chi_u^{\tau\tau}(0,0)\; q_2(0,0) + 2\chi_u^{\tau}(0,0)\; q_2^{\tau}(0,0) +$$

$$+ \chi_u(0,0)\; q_2^{\tau\tau}(0,0)]\,. \qquad \text{(B-4)}$$

Since q_2 is a double Fourier transform of V_2, of exactly the same form as (B-2), it follows immediately, by similarity to (B-3), that the required derivatives of q_2 in (B-4) can be found from smoothing function V_2 as

$$q_2(0,0) = \iint dt\ df\ V_2(t,f)$$

$$q_2^{\nu}(0,0) = -i2\pi \iint dt\ df\ t\ V_2(t,f)$$

$$q_2^{\tau}(0,0) = i2\pi \iint dt\ df\ f\ V_2(t,f)$$

$$q_2^{\nu\nu}(0,0) = -4\pi^2 \iint dt\ df\ t^2\ V_2(t,f)$$

$$q_2^{\nu\tau}(0,0) = 4\pi^2 \iint dt\ df\ t\ f\ V_2(t,f)$$

$$q_2^{\tau\tau}(0,0) = -4\pi^2 \iint dt\ df\ f^2\ V_2(t,f)\ . \qquad \text{(B-5)}$$

COMPLEX AMBIGUITY FUNCTION PROPERTIES

The required derivatives of χ_u in (B-4) can be determined from definition (24). We list them here for completeness and future reference:

$$\chi_u(0,0) = \int dt\ |u(t)|^2$$

$$\chi_u^{\nu}(0,0) = -i2\pi \int dt\ t\ |u(t)|^2$$

$$\chi_u^{\tau}(0,0) = i \int dt\ \mathrm{Im}\{u'(t)\ u^*(t)\}$$

$$\chi_u^{\nu\nu}(0,0) = -4\pi^2 \int dt\ t^2 |u(t)|^2$$

$$\chi_u^{\nu\tau}(0,0) = 2\pi \int dt\ t\ \mathrm{Im}\{u'(t)\ u^*(t)\}$$

$$\chi_u^{\tau\tau}(0,0) = - \int dt\ |u'(t)|^2\ . \qquad \text{(B-6)}$$

These quantities are all real, with the exception of the two single derivatives, both of which are purely imaginary. These second-order derivative values of χ_u can be expressed solely in terms of u(t) and u'(t).

Since we can express complex ambiguity function χ_u in terms of the WDF W_u according to

$$\chi_u(\nu,\tau) = \iint dt\, df\, \exp(-i2\pi\nu t + i2\pi f\tau)\, W_u(t,f) \; , \tag{B-7}$$

it readily follows from (B-6) that

$$\iint dt\, df\, W_u(t,f) = \int dt\, |u(t)|^2$$

$$\iint dt\, df\, t\, W_u(t,f) = \int dt\, t\, |u(t)|^2$$

$$\iint dt\, df\, f\, W_u(t,f) = \frac{1}{2\pi} \int dt\, \mathrm{Im}\{u'(t)\, u^*(t)\}$$

$$\iint dt\, df\, t^2\, W_u(t,f) = \int dt\, t^2\, |u(t)|^2$$

$$\iint dt\, df\, t\, f\, W_u(t,f) = \frac{1}{2\pi} \int dt\, t\, \mathrm{Im}\{u'(t)\, u^*(t)\}$$

$$\iint dt\, df\, f^2\, W_u(t,f) = \frac{1}{4\pi^2} \int dt\, |u'(t)|^2 \; . \tag{B-8}$$

APPENDIX C. GENERAL TILTED ELLIPSE

It will be convenient to be able to specify the area, tilt, and shape factor of an ellipse directly, instead of trying to solve for these quantities from the general form

$$\frac{1}{2}\,ax^2 + \frac{1}{2}\,by^2 + \sqrt{ab}\,\rho xy = 1 \tag{C-1}$$

employed in [1, (J-2)]. Accordingly, as done in [1, app. D], we employ the rotated coordinates depicted in figure C-1 below. The equation of the ellipse in x',y' space is

$$\left(\frac{x'}{x_o'}\right)^2 + \left(\frac{y'}{y_o'}\right)^2 = 1 \ . \tag{C-2}$$

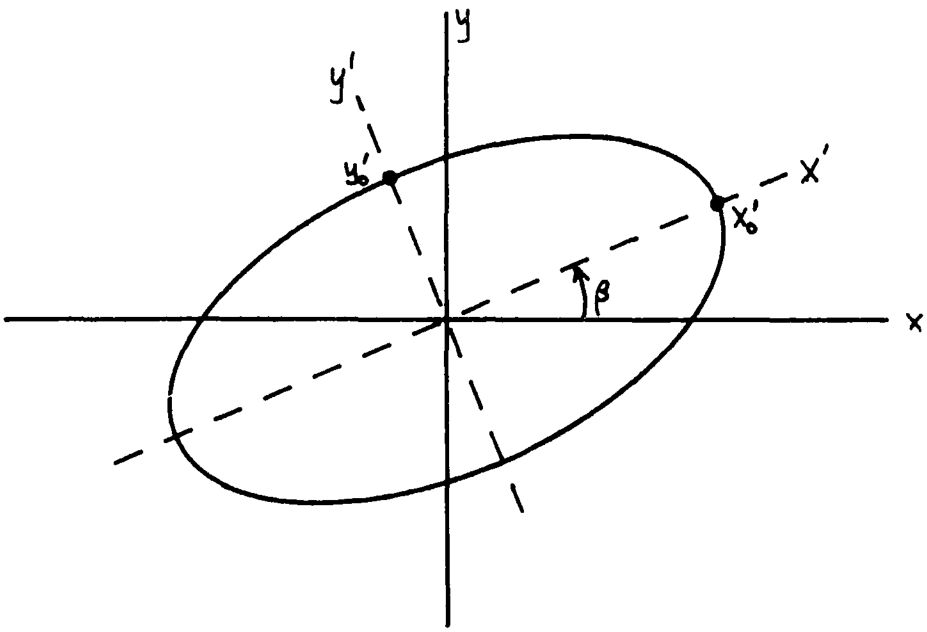

Figure C-1. Rotated Coordinate Axes

But since the area of this ellipse is

$$A = \pi\, x_0' \, y_0' \,, \tag{C-3}$$

while its shape factor is

$$F = \frac{x_0'}{y_0'} \,, \tag{C-4}$$

it is a simple matter to find that

$$A\,F = \pi\, x_0'^2 \,, \quad A/F = \pi\, y_0'^2 \,, \tag{C-5}$$

leading to the desirable form

$$\frac{x'^2}{F} + F\, y'^2 = \frac{A}{\pi} \,. \tag{C-6}$$

Furthermore, the coordinate axes in figure C-1 are related according to

$$\left.\begin{aligned} x' &= xC + yS \\ y' &= -xS + yC \end{aligned}\right\} \quad C = \cos(\beta) \,, \; S = \sin(\beta) \,. \tag{C-7}$$

Substitution in (C-6) yields

$$\frac{1}{2}\, x^2\left(\frac{C^2}{F} + FS^2\right) + \frac{1}{2}\, y^2\left(\frac{S^2}{F} + FC^2\right) + xy\; SC\left(\frac{1}{F} - F\right) = \frac{A}{2\pi} \,, \tag{C-8}$$

which is of the form (C-1) under identifications

$$a = \frac{2\pi}{A}\left(\frac{C^2}{F} + FS^2\right) \,, \quad b = \frac{2\pi}{A}\left(\frac{S^2}{F} + FC^2\right) \,,$$

$$\rho = \frac{\gamma}{\sqrt{1 + \gamma^2}} \quad \text{with} \quad \gamma = SC\left(\frac{1}{F} - F\right) \,. \tag{C-9}$$

Once area A, tilt β, and shape factor F are specified, (C-9) affords a ready calculation of a, b, ρ; quantities C and S are given by (C-7). Since

$\rho = \sin(\Theta)$ in [1, (J-6)], we have

$$\sin(\Theta) = \frac{\gamma}{\sqrt{1 + \gamma^2}} , \quad \cos(\Theta) = \frac{1}{\sqrt{1 + \gamma^2}} , \tag{C-10}$$

which are needed below.

In order to distinguish the two Gaussian mountains being doubly convolved in [1, (J-2)], we label them with subscripts 1 and 2, respectively, thereby obtaining

$$a = \frac{2\pi}{A_1}\left(\frac{C_1^2}{F_1} + F_1 S_1^2\right) , \; b = \frac{2\pi}{A_1}\left(\frac{S_1^2}{F_1} + F_1 C_1^2\right) , \; S_1 = \sin(\beta_1) , \; C_1 = \cos(\beta_1) ,$$

$$\gamma_1 = S_1 C_1 \left(\frac{1}{F_1} - F_1\right) , \; \sin(\Theta) = \frac{\gamma_1}{\sqrt{1 + \gamma_1^2}} , \; \cos(\Theta) = \frac{1}{\sqrt{1 + \gamma_1^2}} , \tag{C-11}$$

and

$$c = \frac{2\pi}{A_2}\left(\frac{C_2^2}{F_2} + F_2 S_2^2\right) , \; d = \frac{2\pi}{A_2}\left(\frac{S_2^2}{F_2} + F_2 C_2^2\right) , \; S_2 = \sin(\beta_2) , \; C_2 = \cos(\beta_2) ,$$

$$\gamma_2 = S_2 C_2 \left(\frac{1}{F_2} - F_2\right) , \; \sin(\phi) = \frac{\gamma_2}{\sqrt{1 + \gamma_2^2}} , \; \cos(\phi) = \frac{1}{\sqrt{1 + \gamma_2^2}} . \tag{C-12}$$

We are now in a position to evaluate the effective area A_3 of the resultant convolution; namely from [1, (J-9)-(J-11)], we have

$$\frac{A_3}{A_1 + A_2} = \frac{\sqrt{D}}{\sqrt{ab}\cos(\Theta) + \sqrt{cd}\cos(\phi)} , \tag{C-13}$$

where

$$D = ab\cos^2(\theta) + cd\cos^2(\phi) + ad + bc - 2\sqrt{abcd}\,\sin(\theta)\sin(\phi)\ . \quad \text{(C-14)}$$

The minimum value of (C-13) is 1, attained when shape factors $F_1 = F_2$ and tilts $\beta_1 = \beta_2$. More generally, when we specify

$$\begin{aligned} &A_1, \beta_1, F_1 \text{ for ellipse } 1\ , \\ &A_2, \beta_2, F_2 \text{ for ellipse } 2\ , \end{aligned} \quad \text{(C-15)}$$

equations (C-11) and (C-12) allow for evaluation of all the parameters needed in (C-13) and (C-14). A sample program in BASIC is attached. Subroutine E computes a, b, $\sin(\theta)$, $\cos(\theta)$ as given by (C-9) and (C-10) in terms of given area A, shape factor F, and tilt β (=B).

```
10    GINIT
20    PLOTTER IS "GRAPHICS"
30    GRAPHICS ON
40    WINDOW -PI/4,PI/4,1,1.25
50    GRID PI/8,.05
60    F1=2                          !  SHAPE FACTOR
70    B1=PI/4                       !  TILT
80    A2=2                          !  AREA
90    F2=2
100   DATA .5,1,2,3,4,5,6
110   DIM A1(1:7)
120   READ A1(*)
130   FOR I=1 TO 7
140   A1=A1(I)
150   CALL E(A1,F1,B1,As,Bs,St,Ct)
160   Ab=As*Bs
170   FOR B2=-PI/4 TO PI/4 STEP PI/100
180   CALL E(A2,F2,B2,Cs,Ds,Sp,Cp)
190   Cd=Cs*Ds
200   D=Ab*Ct*Ct+Cd*Cp*Cp+As*Ds+Bs*Cs-2.*SQR(Ab*Cd)*St*Sp
210   A312=SQR(D)*A1*A2/(2.*PI*(A1+A2))
220   PLOT B2,A312
230   NEXT B2
240   PENUP
250   NEXT I
260   PAUSE
270   END
280   !
290   SUB E(A,F,B,As,Bs,St,Ct)
300   S=SIN(B)
310   C=COS(B)
320   G=S*C*(1./F-F)
330   Sq=SQR(1.+G*G)
340   St=G/Sq
350   Ct=1./Sq
360   C2=C*C
370   S2=S*S
380   T=2.*PI/A
390   As=T*(C2/F+S2*F)
400   Bs=T*(S2/F+C2*F)
410   SUBEND
```

APPENDIX D. KERNEL APPROACH TO PENALTY FUNCTION

The general formulation in (A-11) through (A-21) will be applied in this appendix to the penalty function (3):

$$P(t,f) = a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf \ . \tag{D-1}$$

Substitution in (A-9) (in place of reward R) yields

$$r(t,\tau) = \int df\ \exp(-i2\pi f\tau)\ P(t,f) =$$

$$= \int df\ \exp(-i2\pi f\tau)\ (a^2t^2 + 4\pi^2b^2f^2 + 4\pi ctf) =$$

$$= a^2t^2\delta(\tau) - b^2\delta''(\tau) + i2ct\delta'(\tau) \ . \tag{D-2}$$

Then kernel K follows from (A-11) as

$$K(x,y) = r\left(\frac{x + y}{2}\ ,\ x - y\right) =$$

$$= \frac{a^2}{4}\ (x + y)^2\ \delta(x - y) - b^2\delta''(x - y) + ic\ (x + y)\ \delta'(x - y)\ , \tag{D-3}$$

which is Hermitian.

The integral equation (A-12), that must be solved, can be simplified by use of the facts that

$$\frac{1}{4}\ a^2 \int dx(x + y)^2\ \delta(x - y)\ \phi_n(x) = a^2y^2\phi_n(y)\ ,$$

$$-b^2 \int dx\ \delta''(x - y)\ \phi_n(x) = -b^2\phi_n''(y)\ ,$$

$$ic \int dx\ (x + y)\ \delta'(x - y)\ \phi_n(x) = -ic[2y\phi_n'(y) + \phi_n(y)]\ , \tag{D-4}$$

where the last two results are obtained by integration by parts. Then (A-12) yields differential equation

$$b^2 \phi_n''(y) + i2cy\, \phi_n'(y) + (\lambda_n + ic - a^2y^2)\, \phi_n(y) = 0\,. \tag{D-5}$$

If we try solution

$$\phi_o(y) = A \exp\left(-\frac{1}{2}\, By^2\right) \tag{D-6}$$

in (D-5), we find it to be acceptable if we take

$$B = \frac{\sqrt{Q} + ic}{b^2}\,, \quad \lambda_o = \sqrt{Q}\,. \tag{D-7}$$

These results agree with (11) and (20), as expected. To find the general solution of (D-5), we try solution form

$$\phi(y) = \exp(-\frac{1}{2}\, By^2)\; H(y)\,, \tag{D-8}$$

with B still given by (D-7). This form in (D-8) is <u>no loss of generality</u> since H is still arbitrary. Use of (D-8) in (D-5) results in

$$b^2H''(y) + 2y\, H'(y)\, (-b^2B + ic) +$$

$$+ H(y)\, (-b^2B + b^2B^2y^2 - i2cBy^2 + \lambda + ic - a^2y^2) = 0\,. \tag{D-9}$$

When the value for B in (D-7) is utilized, (D-9) simplifies to

$$b^2H''(y) - 2\sqrt{Q}\, y\, H'(y) + (\lambda - \sqrt{Q}\,)\, H(y) = 0\,. \tag{D-10}$$

(As a partial check, if $H(y) = A$, then $\lambda = \sqrt{Q}$, as in (D-7).)

Now, in (D-9), let

$$H(y) = G(Fy)\,, \quad H'(y) = F\, G'(Fy)\,, \quad H''(y) = F^2\, G''(Fy)\,, \tag{D-11}$$

where F is arbitrary for the moment, thereby obtaining

$$b^2F^2\ G''(Fy) - 2\sqrt{Q}\ y\ F\ G'(Fy) + (\lambda - \sqrt{Q})\ G(Fy) = 0\ . \qquad \text{(D-12)}$$

Now let x = Fy to get

$$G''(x) - \frac{2\sqrt{Q}}{b^2F^2}\ x\ G'(x) + \frac{\lambda - \sqrt{Q}}{b^2F^2}\ G(x) = 0\ . \qquad \text{(D-13)}$$

If we now let (without loss of generality)

$$F = \frac{2^{1/2}Q^{1/4}}{b}\ , \qquad \text{(D-14)}$$

then (D-13) simplifies further to

$$G''(x) - x\ G'(x) + \frac{\lambda - \sqrt{Q}}{2\sqrt{Q}}\ G(x) = 0\ . \qquad \text{(D-15)}$$

We now appeal to [4, 22.6.21] and observe that if

$$\frac{\lambda - \sqrt{Q}}{2\sqrt{Q}} = n = \text{integer}\ , \qquad \text{(D-16)}$$

then a solution of (D-15) is

$$G(x) = He_n(x)\ ,\ \lambda_n = \sqrt{Q}\,(1 + 2n)\ . \qquad \text{(D-17)}$$

Also, (D-11) yields

$$H(y) = G(Fy) = He_n(Fy)\ , \qquad \text{(D-18)}$$

while (D-8) gives

$$\phi_n(y) = A\ \exp(-\tfrac{1}{2}\ By^2)\ He_n(Fy)/\sqrt{n!}\ , \qquad \text{(D-19)}$$

with

$$B = \frac{\sqrt{Q} + ic}{b^2}\ ,\ F = \frac{2^{1/2}Q^{1/4}}{b}\ ,\ |A|^2 = \frac{Q^{1/4}}{\sqrt{\pi}\,b}\ , \qquad \text{(D-20)}$$

where the unit energy normalization of ϕ_n has been imposed. The corresponding eigenvalue follows from (D-17) as

$$\lambda_n = \sqrt{Q}\,(1 + 2n) = \sqrt{a^2b^2 - c^2}\,(1 + 2n) . \qquad \text{(D-21)}$$

The minimum obviously occurs for n = 0.

Result (D-19) agrees with (A-27). However, the λ_n given here by (D-21) differs from that given by (A-26), because we are solving for the minimum penalty here versus the maximum reward there.

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